Ionization Rate Calculations for Preionizers

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The ionization coefficient has been calculated for cesium in argon. The calculation involved numerical solution of the electron Boltzmann equation coupled with the rate equations of the cesium level populations. Results show that resonance radiation escape depresses the high energy portion of the electron distribution function, and reduces the ionization coefficient below its value for no resonance radiation escape. The ionization coefficient is exhibited for a range of values of the electron temperature, electron density, and radiation escape parameter, that is typical of preionizer conditions.

Nomenclature

A_{lk} = effective Einstein A coefficient for spontaneous emission
e = magnitude of electron charge
E = electric field strength
$f(\epsilon)$ = electron distribution function
$f_M(\epsilon)$ = Maxwellian electron distribution function
g_k = statistical weight of the kth level of the atom (hydrogen or cesium)
k = Boltzmann constant
m = electron mass
n_e = free electron number density
$\dot{n}_e$ = dn_e/dt
n_k = number density of atoms in the kth level
$Q_{kl}(\epsilon)$ = energy-dependent excitation (or de-excitation) cross section corresponding to a collisional transition from the kth to the lth level of the atom caused by a free electron
S = ionization coefficient
S_{kl} = rate coefficient describing a collisional transition from the kth to the lth level of the atom
t = time
T_e = electron temperature
T_g = gas temperature
v = electron speed
α = recombination coefficient
β = resonance radiation escape parameter
ϵ = free electron energy
ϵ_0 = permittivity of free space
Λ = ratio of Debye length to average impact parameter
ν_{ee} = electron-electron collision frequency
ν_{eH} = collision frequency of electrons and all heavy particles

Subscripts

h = heavy particle component of gas
$k,l,\ldots$ = index of discrete level of atom ($k = 1$ refers to ground level)
λ = reference to processes involving the free electron continuum

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Introduction

FOR a closed-cycle MHD power generation system operating with a nuclear energy source, the temperature of the gas and the corresponding electric conductivity downstream of the reactor may be insufficient to permit efficient use of the MHD generator. It has been proposed that the gas be preionized before entrance to the generator using, for example, an applied electrical discharge. If the temperature of the gas is not too small, there will be a sufficient density of electrons such that the ionization of the seed will occur primarily as a result of electron-atom collisions.

If this is the case, the significant temperature in the gas is the electron temperature and the preionizer may be viewed as a device to increase the electron temperature of the gas. The electron number density n_e will then tend toward its new higher steady-state value in accordance with the electron energy equation and the macroscopic rate equation

$$\dot{n}_e = n_1 n_e S - n^2{}_e \alpha$$

Here S is the ionization coefficient, α the recombination coefficient, and n_1 the ground state density of the seed. For preionizers the term containing S will predominate because the electron number density is much smaller than its steady-state value. The ionization coefficient S will, in general, depend on the plasma properties.

A simple estimate of the ionization coefficient can be found by applying the method of detailed balancing to the Hinnov and Hirschberg[1] expression for the recombination coefficient. This results in an expression for S which is a function of T_e only (see curve labelled H & H in Fig. 2). Physically this value $S_{H\&H}$ corresponds to the limit in which collisional processes dominate radiative processes. The present work is

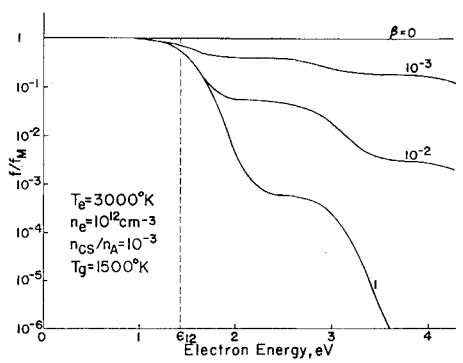

Fig. 1 Electron distribution function normalized on a Maxwellian distribution at the electron temperature.

concerned with the calculation of the ionization coefficient for plasma conditions characteristic of preionizers, where radiative processes may not be neglected.

Rate Equations

Bates, Kingston, and McWhirter[2] have calculated values of S for hydrogen for complete radiation escape (β, the fraction of resonance radiation that escapes the plasma, = 1), and complete resonance radiation trapping (β = 0). Their method is based upon the solution of the quasi-static rate equations for the populations n_k of the excited levels of hydrogen. These rate equations are

$$\dot{n}_k = \sum_{l=1}^{k-1} n_l n_e S_{lk} - n_k \left\{ \sum_{l=1}^{k-1} (n_e S_{kl} + A_{kl}) + \sum_{l=k+1}^{N} n_e S_{kl} + n_e S_{k\lambda} \right\} + \sum_{l=k+1}^{N} n_l (n_e S_{lk} + A_{lk}) + n_e^2 (n_e S_{\lambda k} + A_{k\lambda}) = 0 \quad (1)$$

where

$$S_{kl} = \left(\frac{2}{m}\right)^{1/2} \int_{\epsilon_{kl}}^{\infty} \epsilon f(\epsilon) Q_{kl}(\epsilon) d\epsilon \quad (2)$$

Here A_{kl} denotes the effective Einstein A coefficient for spontaneous emission, and $Q_{kl}(\epsilon)$ is the energy-dependent excitation (or de-excitation) cross section corresponding to a collisional transition from the kth to the lth level. The subscript λ refers to processes involving the electron continuum. These quasi-static equations apply to levels k = 2,3, . . . , N.

Bates, Kingston, and McWhirter used a Maxwellian distribution at the electron temperature

$$f_M(\epsilon) = 2\pi^{-1/2}(kT_e)^{-3/2} \exp(-\epsilon/kT_e)$$

in the calculation of the rate coefficients S_{kl}. Solution of Eq. (1) then yields values of the n_k of the form (for k = 2,3, . . . , N)

$$n_k = n_1 n'_k(n_e, T_e, \beta) + n''_k(n_e, T_e, \beta) \quad (3)$$

where $n'_k(n_e, T_e, \beta)$ and $n''_k(n_e, T_e, \beta)$ are independent of n_1. The equation for the rate of increase of electron density becomes

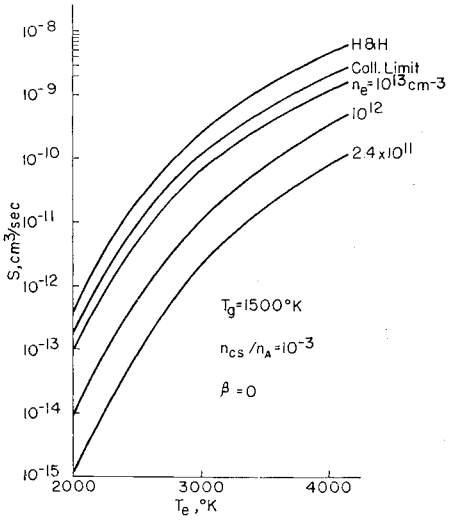

Fig. 2 Ionization coefficient for no radiation escape. The Saha electron density for an electron temperature of 1500°K and seed fraction of 0.1% is 2.4 × 10^{11} cm^{-3}.

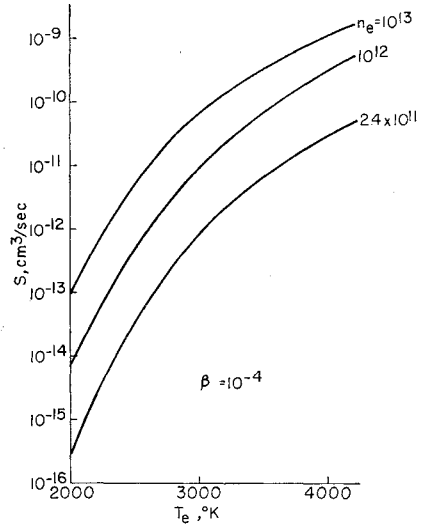

Fig. 3 Ionization coefficient for escape parameter of 10^{-4} and seed fraction (SF) of 0.1%.

$$\dot{n}_e = n_1 n_e \sum_{k=1}^{N} n'_k S_{k\lambda} + n_e \sum_{k=1}^{N} n''_k S_{k\lambda} - n_e^2 \sum_{k=1}^{N} A_{\lambda k} - n_e^3 \sum_{k=1}^{N} S_{\lambda k} \quad (4)$$

Identifying the terms in Eq. (4) that are multiples of n_1 with S and the remaining ones with α, one finds the ionization and recombination coefficients to be

$$S(n_e, T_e, \beta) = \sum_{k=1}^{N} n'_k S_{k\lambda} \quad (5)$$

and

$$\alpha(n_e, T_e, \beta) = \sum_{k=1}^{N} A_{\lambda k} + n_e \sum_{k=1}^{N} S_{\lambda k} - \frac{1}{n_e} \sum_{k=1}^{N} n''_k S_{k\lambda} \quad (6)$$

The parameter β is incorporated into the rate equations through the relation A_{kl} (effective) = βA_{kl} (absolute). Thus, an escape parameter of magnitude less than unity corresponds

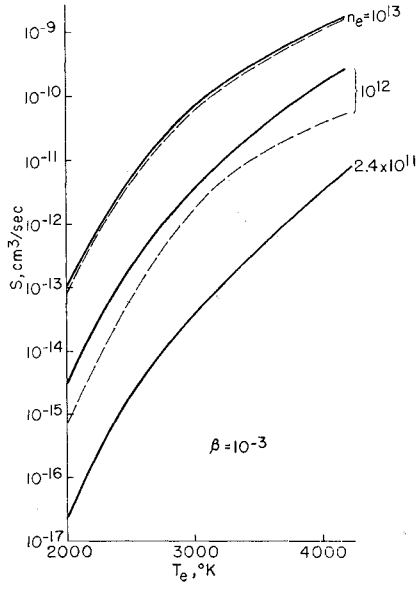

Fig. 4 Ionization coefficient for radiation escape parameter of 10^{-3}. Solid lines correspond to SF = 10^{-3}; dashed lines correspond to SF = 10^{-2}.

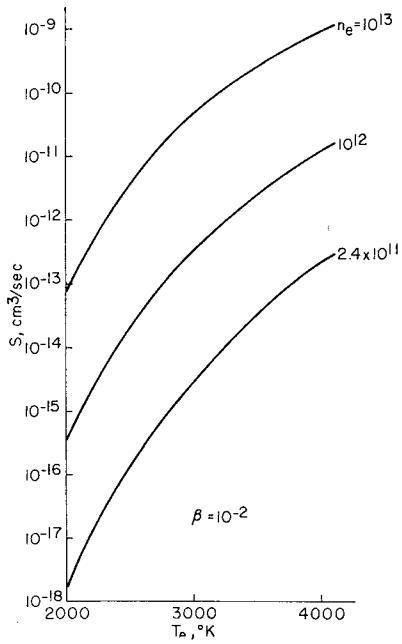

Fig. 5 Ionization coefficient for radiation escape parameter of 10^{-2}, $SF = 10^{-3}$.

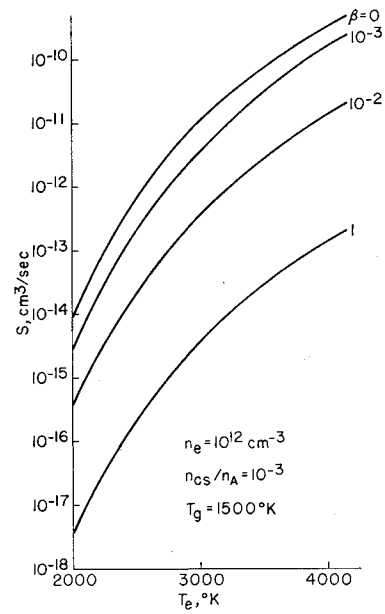

Fig. 7 Variation of the ionization coefficient with radiation escape.

to the situation in which part of the resonance radiation is absorbed before it leaves the plasma. No levels other than the ground level are considered sufficiently populated to absorb line radiation. Continuum radiation to all levels (including the first) is assumed to escape the plasma completely.

Curry, Norcross, and Stone[3] have performed a similar calculation for the ionization coefficient for cesium with $\beta = 0$. As might be expected, the ionization coefficient is most affected by radiation loss when the electron number density is small, which is the case for preionizers.

Electron Distribution Function

The calculations performed in this work include another effect: the depression of the tail of the free electron distribution by nonelastic collisions coupled with radiation

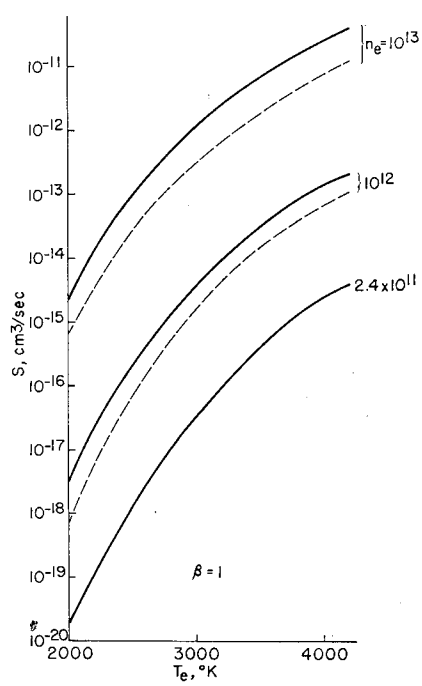

Fig. 6 Ionization coefficient for complete radiation escape. For solid lines, $SF = 10^{-3}$; for dashed lines, $SF = 10^{-2}$.

escape. Because of radiation escape, the rate of inelastic-excitation collisions involving high-energy electrons is not balanced by the inverse superelastic-de-excitation collisional process initiated with low-energy electrons. The net effect is therefore a depletion in the number of high energy electrons (i.e., of electrons with energies $\epsilon > \epsilon_{12}$, where ϵ_{12} is the energy corresponding to the $6S$–$6P$ transition), and a subsequent further depression in the ionization coefficient S (see also Ref. 4).

The values of the ionization coefficient which are presented in this paper are calculated by solving the differential Boltzmann equation including nonelastic collisions coupled with the rate equations for the levels of the cesium atom. The coupling of the Boltzmann and the rate equation arises from the Boltzmann equation nonelastic collision term that is dependent on the n_k and from the fact that the collisional rates in the rate equations are integrals over the distribution function. The solution of these equations was performed numerically using a 20 level model of the cesium atom (which includes the effects of over 100 levels of the actual atom). The nonelastic term in the Boltzmann equation was approximated by retaining only the predominate nonelastic process

$$e + Cs(6S) \rightleftarrows e + Cs(6P)$$

With the assumption of a steady-state and spatial uniformity, the Boltzmann equation may be written[4–7]

$$\frac{d}{d\epsilon}\left[a(\epsilon)\frac{df}{d\epsilon}(\epsilon) + b(\epsilon)\right]f(\epsilon) = C(\epsilon)\left[f(\epsilon) - \frac{n_2}{n_1}\frac{g_1}{g_2}f(\epsilon - \epsilon_{12})\right] \tag{7}$$

where $C(\epsilon) = 0$ for $0 < \epsilon < \epsilon_{12}$, $C(\epsilon) = n_1\epsilon Q_{12}(\epsilon)$ for $\epsilon > \epsilon_{12}$. The coefficients $a(\epsilon)$ and $b(\epsilon)$ are given by

$$a(\epsilon) = \frac{e^2E\epsilon}{3\nu_{eH}/v} + \frac{kT_g\epsilon^2\delta_H\nu_{eH}}{v} + \frac{2\nu_{ee}\epsilon^3}{v}(I_2^0 + J_{-1}^0) \tag{8a}$$

$$b(\epsilon) = \epsilon^2\delta_H\nu_{eH}/v + 3(\nu_{ee}/v)\epsilon^2I_0^0 \tag{8b}$$

where

$$\nu_{eH} = \sum_h \nu n_h Q_h \tag{8c}$$

$$\nu_{ee} = (n_e v/\epsilon^2)(2\pi/3)(e^2/4\pi\epsilon_0)^2 \ln\Lambda \tag{8d}$$

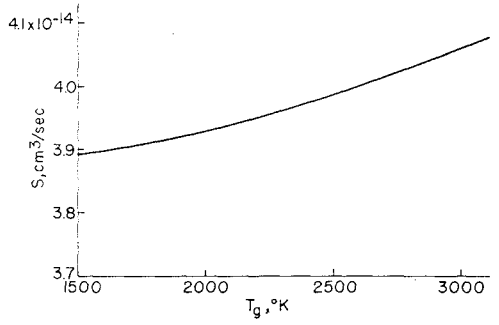

Fig. 8 Variation of the ionization coefficient with gas temperature. $SF = 10^{-3}$, $n_e = 2.4 \times 10^{11}$ cm^{-3}, $T_e = 3000$°K, $\beta = 10^{-3}$, $n_1 = 4.89 \times 10^{15}$ cm^{-3}.

$$\delta_H \nu_{eH} = \sum_h (2m/m_h)\nu n_h Q_h \tag{8e}$$

$$Ip^0 = \frac{1}{\epsilon p^{l/2}} \int_0^{\epsilon} \epsilon^{\frac{1}{2}(p+1)} f(\epsilon) d\epsilon \tag{8f}$$

$$Jp^0 = \frac{1}{\epsilon p^{l/2}} \int_{\epsilon}^{\infty} \epsilon^{\frac{1}{2}(p+1)} f(\epsilon) d\epsilon \tag{8g}$$

The representation of the nonelastic collision term in Eq. (7) is valid if $kT_e \ll \epsilon_{12}$. The average electron energy is defined as $\frac{3}{2}kT_e$, whether the distribution is Maxwellian or not. Numerical solutions of Eq. (7) indicate that the electron distribution function is well represented by the Maxwellian form $\exp(-\epsilon/kT_e)$ in the thermal energy range ($\epsilon < \epsilon_{12}$) if the electric field is not too large (<80 v/cm). This low-energy behavior of the electrons reflects the fact that electron-electron collisions are dominant at these energies. (For small electric fields (<6 v/cm) solutions of Eq. (7) are insensitive to the value of E and solutions may be found corresponding to any specified electron temperature.) For conditions where $f(\epsilon)$ is Maxwellian at thermal energies, the electric field serves primarily to determine the electron temperature, but otherwise has little effect on the distribution function. For this reason it is convenient to take T_e as an independent parameter characterizing the distribution function. Ionization coefficients calculated in this manner, i.e., with T_e specified and no electric field, will in fact be valid to a good approximation for moderate electric fields (<80 v/cm). The relation between T_e and E is provided through the macroscopic energy equation, which is insensitive to electron-electron collisions.

Solutions of the Boltzmann equation for a typical case of cesium in argon with various degrees of radiation escape are presented in Fig. 1. As the escape parameter is decreased towards zero, the distribution function tends toward its Maxwellian form over the entire energy range. Increasing the electron density makes the $a(\epsilon)$ and $b(\epsilon)$ terms in Eq. (7) more significant compared to $C(\epsilon)$; as nonelastic collisions decrease in importance, the electron distribution becomes more nearly Maxwellian.

Ionization Coefficient

Figures 2–7 represent the results of calculations performed for cesium in argon with $T_g = 1500$°K, gas pressure = 1 atm ($n_g = 4.89 \times 10^{18}$ cm^{-3}), and no electric fie'd. There is no resonance radiation escape in the situation of Fig. 2, so the distribution is Maxwellian. Because of radiation loss in nonresonant transitions, the plasma is not in equilibrium; hence the ionization rate is dependent upon the electron density in addition to the electron temperature. The curve labelled collisional limit in Fig. 2 is the asymptote for large electron densities (greater than 10^{14} cm^{-3}). This asymptotic curve will be approached in the limit of large electron number densities for any magnitude of the radiation escape parameter. Figure 6 presents results for complete radiation loss; the depression of the high-energy portion of the electron distribution function together with the reduction of the populations of the excited levels lowers the value of the ionization coefficient to 10^{-5} of its value for no resonance radiation loss. For preionizer conditions in which the escape parameter is intermediate between zero and unity, Figs. 3–5 exhibit the ionization coefficient. In Fig. 7, the effect of variation in radiation escape is demonstrated. The conditions of the plasma for Fig. 7 correspond to those for which the distribution functions of Fig. 1 were calculated.

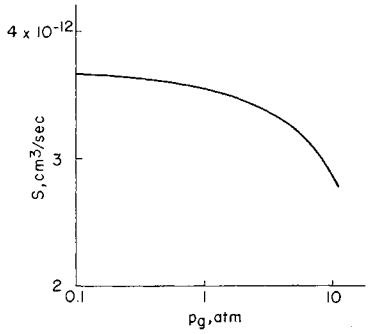

Fig. 9 Variation of the ionization coefficient with gas pressure at constant seed number density. $T_e = 3000$°K, $n_1 = 4.89 \times 10^{15}$ cm^{-3}, $n_e = 10^{12}$ cm^{-2}, $\beta = 10^{-3}$.

The ionization coefficient calculated on the basis of a Maxwellian distribution is a function of three variables only, $S = S(n_e,T_e,\beta)$. If S is determined from a non-Maxwellian distribution function, then the ionization coefficient will depend also on the variables that affect $f(\epsilon)$. From Eqs. (7) and (8) and the discussion following Eq. (8), $f = f(\epsilon; T_e,n_e,n_1,n_g,E)$, so the functional dependence of S must now be described as $S = S(n_e,T_e,\beta;\ n_1,n_g,T_g,E)$. The variables to the left of the semicolon have a major influence on the ionization coefficient. The relative insensitivity of S on n_g,T_g, and E is shown in Figs. 8–10, respectively. The dependence on n_1 (through the seed fraction n_1/n_g) is exhibited in Figs. 4 and 6.

Conclusions

The electron distribution function solution to the Boltzmann equation including nonelastic collisions has been shown to have a Maxwellian form for electron energies smaller than ϵ_{12}, the energy of the first excited level. If resonance radiation is considered to partially or completely escape the plasma, the high-energy ($>\epsilon_{12}$) tail of the distribution function may be severely depressed. The effect decreases the ionization coefficient even more than would be calculated on the basis of the theory of Bates, Kingston, and McWhirter. The radiation escape parameter thus assumes as important a role as the electron density and temperature in determining the ionization coefficient.

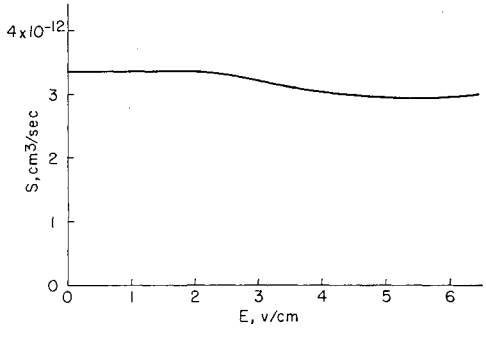

Fig. 10 Variation of the ionization coefficient with electric field. $SF = 10^{-3}$, $n_e = 2.4 \times 10^{11}$ cm^{-3}, $\beta = 10^{-3}$, $n_1 = 4.89 \times 10^{15}$ cm^{-3}, $T_g = 1500$°K.

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A Terminal Guidance Theory Using Dynamic Programing Formulation

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The dynamic programing condition for minimum error (cost function) for a general η dimensional dynamic process is extended to optimization problems subject to integral-type constraints. The generalized dynamic programing formulation is then applied to the implicit terminal guidance problems. An optimal guidance law in the neighborhood of a nominal (optimal or not) trajectory is developed. This guidance law is optimal in the sense of guiding a vehicle from an arbitrary starting point to an arbitrary fixed-end point along a neighborhood trajectory with minimum control energy consumption. It is not approximate optimization in the same sense of the nominal trajectory if an optimal trajectory in some sense is used as nominal. The resulting guidance law is a closed-loop feedback control in terms of state variables and Lagrange multipliers. The equations for computing the Lagrange multipliers to satisfy the terminal constraints are derived. Thus, a control law in terms of only the terminal constraints, initial conditions, and state variables is obtained. This control law also can be applied to optimization and control of nonlinear systems and solving two-point boundary problems.

I. Introduction

THE problem of terminal guidance has been studied previously by Kelley,[1,2] Breakwell, Bryson, and Spener.[3,4] In all these references, an optimized nominal trajectory resulting from variational calculus, which minimizes (or maximizes) a function of certain terminal quantities while satisfying specified initial and terminal conditions, was used as a reference trajectory. The properties of the optimal nominal trajectory were employed to develop a feedback guidance law that is approximately optimal in the same sense as the nominal trajectory. The terminal quantities being controlled or optimized vary with initial conditions and control scheme used. The guidance schemes developed in these references fail in cases in which the nominal trajectory is not optimal and all the terminal quantities are specified. Furthermore, to compute the optimal nominal trajectory is a difficult two-point boundary-value problem. To avoid the difficulty of obtaining an optimal trajectory and to solve the guidance problem that all the terminal quantities are specified, a terminal guidance theory with a different philosophy is developed in this paper.

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Here, implicit guidance philosophy is used and any obtainable trajectory (not necessarily optimal) between two end points and the associated control policy are used as nominal trajectory and reference control. The guidance problem treated here is to guide a vehicle from the initial point region to the terminal point region along the neighborhood of the nominal trajectory with minimum control effort. All the terminal values of the state variables are specified (a fixed-end point problem), and a cost function related to the control effort is set up to be minimized. The optimal control law is derived using dynamic programing formulation to provide zero terminal errors by using the desired terminal conditions as terminal constraints in the optimization process.

The design of optimum control systems using dynamic programing formulation has been treated fully in the literature. The dynamic programing condition for minimum error for a general η dimensional dynamic process is a partial differential equation of the minimum error (cost) function,[5] and leads to free-point terminal-boundary conditions. In Ref. 5, the minimum error function is assumed to be a quadratic form of the state variables of the system, and thus results in a feedback optimal control law that is linear in the state variables.

Optimization problems subject to an integral constraint for a first-order dynamic process also were treated in Ref. 5. This can introduce fixed-point terminal conditions into the dynamic programing formulation of the condition for mini-